1987

Design of 2-D FIR filters with nonuniform frequency samples

Diamessis, John E.

DESIGN OF 2-D FIR FILTERS WITH NONUNIFORM FREQUENCY SAMPLES

John E. Diamessis*, Charles W. Therrien, and William J. Rozwod

Naval Postgraduate School
Monterey, California, 93943

Abstract

A method of designing two-dimensional (2-D) FIR filters, using nonuniform frequency samples, is presented. The method is based on an extension of the DFT method of design, which uses uniform frequency samples. The proposed method, is based on an extension of Newton's interpolation method to 2-D. The proposed procedure has the attractive properties of permanence and recursive computation of the design parameters. A design procedure is given for 2-D FIR linear phase filters and an example is given.

INTRODUCTION

Because of the fundamental nature of the interpolation process, methods using combinations of various interpolating functions have been used extensively in numerical mathematics [1], electrical engineering in general [2], and digital signal processing in particular [3-6]. In this paper we propose a nonuniform frequency sampling method for designing 2-D FIR digital filters. FIR filter design methods are covered in [7,8]. The FIR design method generally known as "frequency sampling" is based on the DFT and therefore is applicable to uniform frequency samples only [7-10]. The method proposed here is based on Newton's interpolation method and can be applied to both uniform and nonuniform samples. In this sense, it is a generalization of the DFT method.

Newton's method also has other desirable characteristics, namely, recursive computation of the design parameters and the permanence property. By permanence we mean the ability of the method to use more frequency samples to design higher order filters without having to recompute the "old" design parameters. From a computational point of view, the design procedure is reduced to the solution of two triangular systems of linear equations.

Although the methods to be described are applicable in both 1 and 2 dimensions we shall discuss only the 2-D case here. The 1-D case has been considered earlier by Schüssler [11] and the reader can refer to that publication for details. The 2-D case is much more interesting since choosing the frequency response samples and generating the Newton form of the filter is not so straightforward.

NEWTON INTERPOLATION IN 2-D

We describe below Newton's interpolation method for the case of a 2-D quadratic polynomial [4,10]. The extension of the method to higher degree 2- D polynomials and higher dimensions is straightforward, although the algebraic expressions become more complex. The 2-D quadratic polynomial interpolation problem can be stated as follows:

Given:

1) a set of points in the plane:

$$\mathbf{S}_f \triangleq \{(x_0,y_0),(x_0,y_1),(x_0,y_2),(x_1,y_0),(x_1,y_1),(x_2,y_0)\} \tag{1}$$

2) a set of values at those points:

$$\mathbf{F} \triangleq \left\{ \begin{array}{l} f(x_0,y_0) \triangleq f_{00},\ f(x_0,y_1) \triangleq f_{01},\ f(x_0,y_2) \triangleq f_{02}, \\ f(x_1,y_0) \triangleq f_{10},\ f(x_1,y_1) \triangleq f_{11},\ f(x_2,y_0) \triangleq f_{20} \end{array} \right\} \tag{2}$$

3) a quadratic 2-D polynomial:

$$p(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \tag{3}$$

find

$$\underline{a} \triangleq [a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02},]^T \tag{4}$$

such that the following interpolating conditions are satisfied

$$\left. \begin{array}{lll} p(x_0,y_0) = f_{00}, & p(x_0,y_1) = f_{01}, & p(x_0,y_2) = f_{02} \\ p(x_1,y_0) = f_{10}, & p(x_1,y_1) = f_{11}, & p(x_2,y_0) = f_{20} \end{array} \right\} \tag{5}$$

* On leave National Technical University of Athens.

The basic idea in solving this problem by Newton's method is to write $p(x, y)$ in (3) in the form

$$
\begin{aligned}
p(x,y) = c_{00} &+ c_{10}(x - x_0) + c_{01}(y - y_0) \\
&+ c_{20}(x - x_0)(x - x_1) + c_{11}(x - x_0)(y - y_0) \\
&+ c_{02}(y - y_0)(y - y_1)
\end{aligned}
\tag{6}
$$

If we define

$$\underline{c} \triangleq [c_{00}, c_{10}, c_{01}, c_{20}, c_{11}, c_{02}]^T \tag{7}$$

The coefficient vectors $\underline{a}$ and $\underline{c}$ in the polynomials (3) and (6) are related by

$$\underline{c} = U_{2D}\underline{a} \tag{8}$$

where U_{2D} is a unit upper triangular matrix [10, 12]. Applying then the conditions (5) to (6) we obtain

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & x_1 - x_0 & 0 & 0 & 0 & 0 \\
1 & 0 & y_1 - y_0 & 0 & 0 & 0 \\
1 & x_2 - x_0 & 0 & (x_2 - x_0)(x_2 - x_1) & 0 & 0 \\
1 & x_1 - x_0 & y_1 - y_0 & 0 & (x_1 - y_0)(y_1 - y_0) & 0 \\
1 & 0 & y_2 - y_0 & 0 & 0 & (y_2 - y_0)(y_2 - y_1)
\end{bmatrix}
\begin{bmatrix} c_{00} \\ c_{10} \\ c_{01} \\ c_{20} \\ c_{11} \\ c_{02} \end{bmatrix}
=
\begin{bmatrix} f_{00} \\ f_{10} \\ f_{01} \\ f_{20} \\ f_{11} \\ f_{02} \end{bmatrix}
\tag{9}
$$

or

$$\underline{N}_{2D}\underline{c} = \underline{f} \tag{10}$$

with the obvious definitions of $\underline{N}_{2D}$ and $\underline{f}$. We also have that

$$|\underline{N}_{2D}| = (x_1 - x_0)^2(y_1 - y_0)^2(x_2 - x_0)(x_2 - x_1)(y_2 - y_0)(y_2 - y_1) \tag{11}$$

and $|N_{2D}| \neq 0$, provided x_i are distinct and y_i are distinct. Existence and uniqueness can be easily established from (9) and $\underline{N}_{2D}^{-1}$ can be computed. Because of the triangular nature of (9) the c's can be computed recursively and have the permanence property [4,10].

Combining (10) with (8) we obtain

$$\underline{N}_{2D}\underline{U}_{2D}\underline{a} = \underline{f} \tag{12}$$

which can be solved for $\underline{a}$. Computationally, it is more efficient to determine $\underline{a}$ by solving the two triangular systems (10) and (8). It can be shown that the matrices N_{2D} and U_{2D} are the factors of the "LU" decomposition of the matrix of the system which results from direct application of the conditions (5) to (3) [10, 12]. There are other methods for solving interpolation problems. However, Newton's method has the advantages of recursive computability and permanence, and will be used in the sequel.

FILTER DESIGN METHOD

In the 2-D FIR filter design problem one is given

(1) A set of samples (the *support set*)

$$\mathbb{Z} = \left\{(z_{1k}z_{2k}) = (e^{j\omega_{1k}}, e^{j\omega_{2k}}), k = 0, 1, 2, \ldots\right\} \tag{13}$$

(2) A set of values at those points

$$\mathbb{H} = \{H(z_{1k}, z_{2k}) = H_k, \quad k = 0, 1, 2, \ldots\} \tag{14}$$

(3) A filter of the form

$$H(z_1, z_2) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} h(n_1, n_2) z_1^{-n_1} z_2^{-n_2} \tag{15}$$

The problem is then to find the $h(n_1, n_2)$ such that

$$\mathbb{H}(e^{j\omega_{1k}}, e^{j\omega_{2k}}) = H_k, \quad k = 0, 1, 2, \ldots \tag{16}$$

Note that this problem, as stated, is essentially an interpolation problem and that the given frequency samples are not required to be uniformly spaced. However in 2-D there are certain other topological constraints on the support set in order to generate a Newton form and therefore a filter of fixed order. A discussion of the possible support sets is given in [4]. For present purposes however it will be assumed that $N_1 = N_2 = N$ and that $(N+1)(N+2)/2$ frequency samples are chosen of the form

$$\{((\omega_{1k_1}\omega_{2k_2}), k_2 = 0, 1, \ldots, k_1), k_1 = 0, 1, 2, \ldots N\} \tag{17}$$

A support set of this form will be called *triangular* and it insures that a Newton form representation with a polynomial of degree N exists. Note however that since there is no constraint such as $\omega_{i0} < \omega_{i1} < \omega_{i2} \ldots$ the points selected may not appear in a geometrically triangular region of the $\omega_1\omega_2$ plane (see Fig. 1 in the example below).

A brief outline of the filter design method for 2-D FIR linear phase filters is given below. Space limitations prevent more detailed derivation of results. To guarantee the linear phase property the symmetry condition

$$h(n_1, n_2) = h(N - 1 - n_1, N - 1 - n_2) \tag{18}$$

is applied to (15) and the result is evaluted at $z_1 = e^{j\omega_1}, z_2 = e^{j\omega_2}$. After some algebraic manipulation, this can be put in the form

$$\mathrm{H}(e^{j\omega_1}, e^{j\omega_2}) = h(M,M) + 2\sum_{k_2=0}^{M-1} h(M,k_2)T_{M-k_2}(\cos\omega_2)$$
$$+ 2\sum_{k_1=0}^{M-1}\sum_{k_2=0}^{2M} h(k_1,k_2)(\cos w_1)T_{M-k2}(\cos\omega_2)$$
$$- 2\sum_{k_1=0}^{M-1}\sum_{k_2=0}^{2M} h(k_1,k_2)\sin(\omega_1(M-k_1))\sin(\omega_2(M-k_2)) \tag{19}$$

where $M = \frac{N-1}{2}$ and $T_n(\cdot)$ is the nth Chebyshev polynomial. If the further symmetry condition

$$h(n_1,n_2) = h(n_1, N_2 - 1 - n_2) \tag{20}$$

is applied the terms involving the sin become zero. Then with the transformations

$$\cos\omega_1 = 1 - 2x \tag{21a}$$
$$\cos\omega_2 = 1 - 2y \tag{21b}$$

the resulting polynomial can be put in the Newton form

$$G(x,y) = c_{00} + \sum_{i=1}^{M} c_{i0}\sum_{p=0}^{i-1}(x - x_p) + \sum_{j=1}^{M} c_{0j}\prod_{q=0}^{j-1}(y - y_q)$$
$$+ \sum_{i=1}^{M}\sum_{j=1}^{M} c_{ij}\prod_{p=0}^{i-1}(x - x_p)\prod_{q=0}^{j-1}(y - y_q) \tag{22}$$

Methods discussed in the previous section can then be used to find the coefficients c_{ij}. This results in a specific structure and frequency response for the filter.

EXAMPLE

This example illustrates the design of a 2-D low-pass filter. A support set of 136 ($M = 15$) points in the form (17) chosen to sample the ideal frequency response in the critical passband and stop band areas is shown in Fig. 1. Coefficients in the Newton representation (22) are found by forming and solving equations (9). The frequency response evaluated using (21) and (22) is shown in Fig. 2. The design achieves flat response in the pass band, sharp cutoff, and low ripple in the stop band with a relatively small number of points.

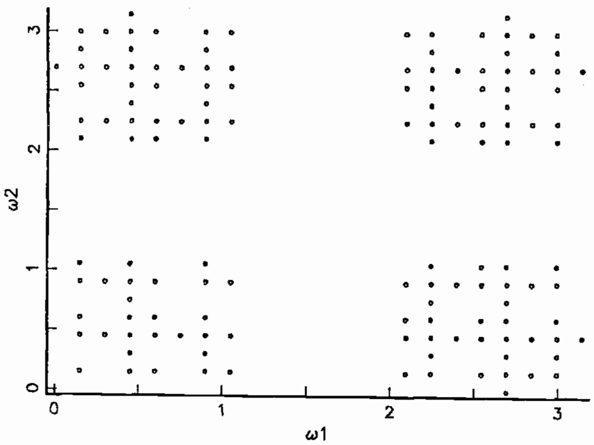

Fig. 1 Frequency samples for design of a 2-D low pass filter.

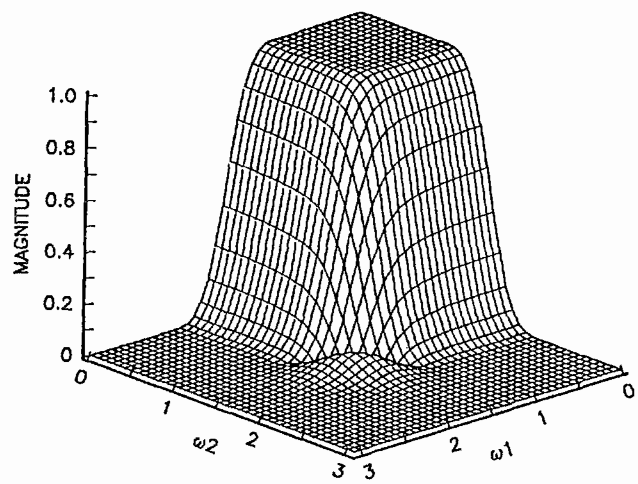

Fig. 2 Low pass filter magnitude.

CONCLUSIONS

A method for design of 2-D filters based on 3-D interpolation was described. Similar but considerably simpler methods for 1-D filters are available. In 2-D the form of the frequency sampling, i.e., the support set, is critical. However since the sampling is non-uniform there is considerable flexibility in choosing the points to achieve a desired frequency response characteristic. An example of a low pass filter design problem showed how a triangular support set was applied in practice. The design based on a relatively small set of frequency samples showed excellent frequency response in the critical regions.

Acknowledgement

This work was supported by the Naval Postgraduate School Foundation research program. This support is gratefully acknowledged.

References

1. C. E. Fröborg, Numerical Mathematics and Computer Applications, Benjamin, 1985.
2. R. N. Bracewell, The Fourier Transform and Its Applications, McGraw-Hill, 2nd Ed., 1986.
3. M. Bellanger, Digital Processing of Signals, Wiley, 1984. (translated from the French).
4. J. E. Diamessis and C. W. Therrien, "Modeling Unequally Spaced 2-D Discrete Signals by Polynomials," Proceedings of the 20th Asilomar Conference on Signals, Systems and Computers, Pacific Grove, CA, Nov. 1986.
5. C. S. Williams, Designing Digital Filters, Prentice-Hall, 1986.
6. A. J. E. M. Jensen *et.al.*, "Adaptive Interpolation of Discrete-Time Signals that can be Modeled as Autoregressive Processes," IEEE Trans. ASSP, pp. 317-330, April 1986.
7. L. R. Rabiner and B. Gold, Theory and Application of Digital Signal Processing, Prentice-Hall, 1975.
8. D. E. Dudgeon and R. M. Mersereau, Multidimensional Digital Signal Processing, Prentice-Hall, 1984.
9. R. R. Bitmead and B. D. O. Anderson, "Adaptive Frequency Sampling Filters," IEEE Trans. on Circuits & Systems, June 1981.
10. J. E. Diamessis, Multidimensional Interpolation Problems with Applications to Multidimensional Signal Processing, Technical Report, Naval Postgraduate School, Aug. 28, 1986.
11. W. Schüssler, "On Structures for Nonrecursive Digital Filters," Arch. Elek. Übertragung, Vol. 26, pp. 255-258, June 1972.
12. J. E. Diamessis and C. W. Therrien, "On the Relation Between Triangular Matrix Factorization and 2-D Polynomial Interpolation," Proc. IEEE ISCAS Conf., May 1987 (to appear).